CHAPTER 6 : CIRCULAR MOTION

INTRODUCTION

- Circular motion is a movement of an object along the circumference of a circle or rotation along a circular path.
- It can be uniform, with constant angular rate of rotation and constant speed, or non-uniform with a changing rate of rotation.
- The rotation around a fixed axis of a three-dimensional body involves circular motion of its parts.
- Examples of circular motion
  - (I) an artificial satellite orbiting the Earth at a constant height,
  - (II) a stone which is tied to a rope and is being swung in circles,
  - (III) a car turning through a curve in a race track,
  - (IV) an electron moving perpendicular to a uniform magnetic field

ELEMENTS OF CIRCULAR MOTION

TIME PERIOD (T)

- It is the time taken to complete one revolution on circular path.

FREQUENCY (ν)

- It is the number of revolutions performed in one second.

  Unit : $\sec^{-1}$ or Hertz or rps (revolution per second)
  1 rpm (revolutions per minute) = 1/60 rps

LINEAR DISPLACEMENT ($d\vec{S}$)

- It is the displacement of body on circular track.
- It is directed tangentially along the direction of motion at any instant and its direction changes continuously.

ANGULAR DISPLACEMENT ($d\vec{\theta}$)

- It is the angle subtended by the body at centre during circular motion.
- Its SI unit is radian.
- It is axial vector and its direction is found by right hand rule.
- The direction of angular displacement does not change at all.

- Relation between linear displacement and angular displacement is given as :

  $d\theta = \frac{ds}{r}$ $\qquad$ $ds = r \times d\theta$

  In vector form - $\boxed{d\vec{s} = d\vec{\theta} \times \vec{r}}$ $\qquad$ $\vec{r}$ = radius vector which is directed radially outward

DIFFERENT VELOCITIES OF BODY

(1) LINEAR VELOCITY ($\vec{v}$) **: -**

- The rate of change of linear displacement is called as linear velocity.

$$\vec{v} = \frac{d\vec{s}}{dt} \text{ m/s}$$

- Its direction is tangential which surely changes continuously.
- The magnitude of linear velocity i.e. linear speed may or may not change.

(2) ANGULAR VELOCITY OR ANGULAR FREQUENCY $(\vec{\omega})$:

- The rate of change of angular displacement is called as angular velocity.

$$\vec{\omega} = \frac{d\vec{\theta}}{dt} \text{ rad / sec}$$

- If the body is moving with uniform angular velocity.

$$\boxed{\vec{\omega} = \frac{2\pi}{T} = 2\pi\nu} \text{ rad/sec}$$

- Angular velocity is a vector quantity and its direction is also found by right hand rule same as angular displacement.
- The direction of angular velocity does not change in circular motion, but its magnitude may or may not change.
- The magnitude of angular velocity (i.e. angular speed) changes only when the magnitude of linear velocity (i.e. linear speed) changes.
- The relation between linear velocity and angular velocity is given as :

$$\because \ d\theta = \frac{ds}{r} \qquad \frac{d\theta}{dt} = \frac{ds}{dt \times r} \qquad \omega = \frac{v}{r}$$

$$\boxed{v = r\omega}$$ In vector from - $\boxed{\vec{v} = \vec{\omega} \times \vec{r}}$

DIFFERENT ACCELERATION OF BODY

- The body possess linear acceleration as well as angular acceleration (may or may not be).

(1) LINEAR ACCELERATION $(\vec{a})$ **:** This acceleration have following two components -

(A) Centripetal / Radial acceleration $(\vec{a}_c)$:

- It is the component which exists due to change in the direction of linear velocity only.
- It is perpendicular to direction of motion, directed towards center i.e. radially inward.
- Its direction changes continuously and its magnitude is given as

$$\boxed{|\vec{a}_c| = \frac{v^2}{r} = r\omega^2} \text{ m/s}^2$$

(B) Tangential acceleration $(\vec{a}_T)$:

- This component exists due to change in the magnitude of linear velocity only.
- It is directed tangentially either in the direction of velocity or in just opposite direction of velocity.
- Its magnitude is given as

$$\boxed{|\vec{a}_T| = \frac{d|\vec{v}|}{dt} = \frac{dv}{dt}} \text{ m / sec}^2$$

- The net acceleration of the body is given as :

$$\boxed{\vec{a} = \frac{d\vec{v}}{dt} = \vec{a}_C + \vec{a}_T} \qquad \boxed{a = \sqrt{a_C^2 + a_T^2}}$$

(2) ANGULAR ACCELERATION $(\vec{\alpha})$:

- This acceleration exists due to change in the magnitude of angular velocity only.
- Its direction is also axial and its magnitude is given as ;

$$\boxed{\vec{\alpha} = \frac{d|\vec{\omega}|}{dt} = \frac{d\omega}{dt}} \text{ rad / sec}^2$$

- Angular acceleration 'α' is concerned with tangential acceleration a_T, because magnitude of angular velocity changes only when magnitude of linear velocity changes. But 'a_c' and 'a_T' or 'a_c' and 'α' are not concerned.

$v = r\omega \qquad \frac{dv}{dt} = r \times \frac{d\omega}{dt} \qquad \boxed{a_T = r\alpha} \qquad \text{in vector form : } \boxed{\vec{a}_T = \vec{\alpha} \times \vec{r}}$

- The equation of motion for linear motion and angular motion are as follows :

$v = u + a_T t \qquad \omega = \omega_0 + \alpha t$

$S = ut + \frac{1}{2}a_T t^2 \qquad \theta = \omega_0 t + \frac{1}{2}\alpha t^2$

$v^2 = u^2 + a_T S \qquad \omega^2 = \omega_0^2 + 2\alpha\theta$

S = length on circular track $\qquad \omega_0$ = initial angular velocity

θ = total angle traversed $\qquad \omega$ = final angular velocity

DIFFERENT FORCES ON BODY

- The net force on body have two components one is radial component (must) and second is tangential component (may or may not be).

(A) Centripital force/ Radial force ($\vec{F}_C$) :

- Due to this component there is change in the direction of velocity only.
- This component of force is responsible to produce centripital acceleration (a_c).
- It is directed radially inward and the direction changes continuously.
- Its magnitude is given as

$F_C = ma_C \qquad \boxed{F_C = \frac{mv^2}{r} = mr\omega^2}$

- $\frac{mv^2}{r}$ is not a new force, but it is a requirment which is full filled by some real force, like gravitational force, electric force, tension etc i.e. these real forces behave as centripital force.

(B) Tangential force ($\vec{F}_T$) :

- Due to this component of force the magnitude of linear velocity as well as angular velocity changes.
- This component of force is responsible to produce tangential acceleration (a_T) as well as angular acceleration (α).
- Its direction is tangential and its magnitude is given as

$\boxed{F_T = ma_T = \frac{mdv}{dt}} \qquad \text{also, } \boxed{F_T = mR\alpha}$

- The net force on the body is given as :

$\boxed{\vec{F} = \vec{F}_C + \vec{F}_T} \qquad \boxed{F = \sqrt{F_C^2 + F_T^2}}$

Circular motion

Uniform circular motion (UCM) | **Non-uniform circular motion**

STUDY OF UNIFORM CIRCULAR MOTION

- Magnitude of linear velocity or linear speed $|\vec{v}|$ = constant
- Linear velocity, $\vec{v}$ = variable (due to change in direction changes)

- Angular velocity, $\vec{\omega}$ = constant (neither magnitude nor direction changes)
- Different possible acceleration :

$$\vec{a}_C \neq 0 , \quad \vec{a}_T = 0 , \quad \alpha = 0$$

- Net linear acceleration $\vec{a} = \vec{a}_c$ (variable as direction changes)
- Since, $\vec{F}_C \neq 0$, $\vec{F}_T = 0$ Net force $\vec{F} = \vec{F}_c$ (variable as direction changes)
- Kinetic energy, K= constant , therefore the change in kinetic energy $\boxed{\Delta K = 0}$
- Workdone by force $W = \int \vec{F} \bullet d\vec{s} = \int F_C \, ds \cos 90^0 = 0$
- Power delivered by force, $P = \vec{F} \bullet \vec{v} = F_C v \cos 90^0 = 0$

STUDY OF NON-UNIFORM CIRCULAR MOTION

- Magnitude of linear velocity or linear speed $|\vec{v}|$= variable
- Linear velocity $\vec{v}$ = variable
- Angular velocity, $\vec{\omega}$ = variable (only magnitude changes not the direction)
- Different possible accelerations :

$$\vec{a}_C \neq 0 , \quad \vec{a}_T \neq 0 , \quad \alpha \neq 0$$

- Net acceleration $\vec{a} = \vec{a}_C + \vec{a}_T$, $|\vec{a}| = \sqrt{a_c^2 + a_T^2}$
- Since, $\vec{F}_C \neq 0$, $\vec{F}_T \neq 0$
- Net force $\vec{F} = \vec{F}_C + \vec{F}_T$, $|\vec{F}| = \sqrt{F_C^2 + F^2{}_T}$
- Kinetic energy, K = variable, $\boxed{\Delta K \neq 0}$
- Workdone by force $= \vec{F} \bullet d\vec{s} = \int (\vec{F}_c + \vec{F}_T).d\vec{s} = \int \vec{F}_c .d\vec{s} + \int \vec{F}_T .d\vec{s}$

$$W = \int \vec{F}_T . d\vec{s} \neq 0 \qquad (\vec{F}_C . d\vec{s} = 0 \text{ because } \vec{F}_C \perp d\vec{s})$$

- Power delievered by force , $P = \vec{F} \bullet \vec{v} = (\vec{F}_C + \vec{F}_T) \bullet \vec{v} = \vec{F}_C . \vec{v} + \vec{F}_T . \vec{v} = \vec{F}_C \vec{v} \cos 90^0 + \vec{F}_T \vec{v}$

$$\boxed{P = \vec{F}_T . \vec{v}}$$

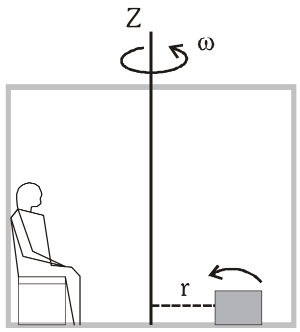
- Clearly during non-uniform circular motion tangential component of force is responsible for workdone and power delievered.

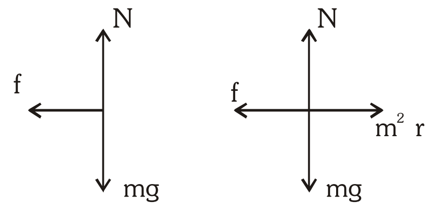
CENTRIFUGAL FORCE

- Centrifugal force is a sufficient pseudo force, only if we are analysing the particles at rest in a uniform rotating frame.
- It is a common misconception that centrifugal force acts on a particle because the particle goes on a circle.
- Centrifugal force acts (or is assumed to act) because we describe the particle from a rotating frame which is nonintertial and still use Newton's laws i.e. centrifugal force is observed when circular motion is seen from non-inertial frome.
- Centrifugal force acts radially outward and its magnitude is equal to centripetal force.

EXAMPLE

- Suppose the observer is sitting in a closed cabin which is made to rotate about the vertical Z-axis at a uniform angular velocity ω. The X and Y axes are fixed is the cabin. Consider a heavy box of mass m kept on the floor at a distance r from the Z-axis. Suppose the floor and the box are rough and the box does not slip on the floor as the cabin rotates. The box is at rest with reslpect to the cabin and hence is rotating with respect to the ground at an angular velocity ω.





OBSERVATION I OBSERVATION II

OBSERVATION I :

- First analyse the motion of the box from the ground frame. In this frame (which is inertial) the box is moving in a circle of radius r.
  - (a) It, therefore, has an acceleration $\frac{v^2}{r} = \omega^2 r$ towards the centre.
  - (b) The resultant force on the box must be towards the centre and its magnitude must be $m\omega^2 r$.
- The forces on the box are
  (i) Weight (mg) (ii) Normal force (N) by the floor (iii) Friction (f) by the floor

OBSERVATION II :

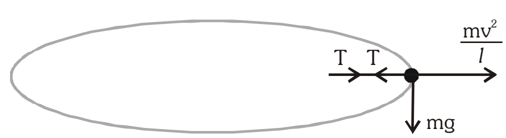
- Now consider the same box when observed from the frame of the rotating cabin(non-intertial frame). The observer there finds that the box is at rest. If he applies Newton's laws, the resultant force on the box should be zero. The analysis from the rotating frame is as follows :
  - (a) The weight and the normal contact force balance each other but the frictional force $f = m\omega^2 r$ acts on the box towards the origin.
  - (b) To make the resultant zero, a pseudo force must be assumed which acts on the box away from the centre (radially outward) and has a magnitude $m\omega^2 r$. This pseudo force is called the centrifugal force.
- The forces on the box are
  (i) Weight (mg) (ii) Normal force (N) (iii) Friction (f) (iv) Centrifugal force ($m\omega^2 r$)
- It any body is moved on circular path without providing the required centripetal force, then it moves radially outward.

CIRCULAR MOTION IN HORIZONTAL PLANE

Example I :

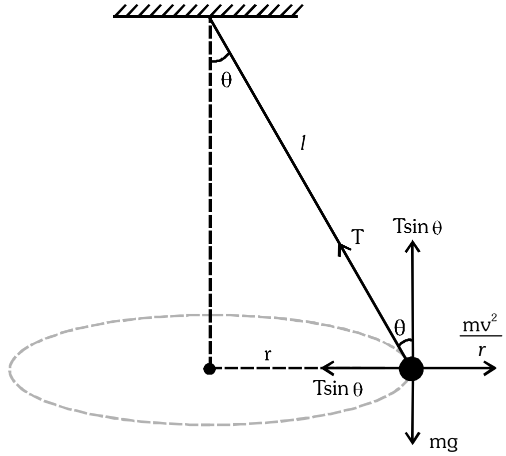
MOTION OF A BODY TIED WITH A STRING

- A body is whirled with the help of massless string on horizontal plane, then



$$T = \frac{mv^2}{l} = ml\omega^2$$

CONICAL PENDULUM



$T\sin\theta = \frac{mv^2}{r} = mr\omega^2$ $T\cos\theta = mg$ $\frac{T\sin\theta}{T\cos\theta} = \frac{mv^2}{r \times mg}$

$$\boxed{\tan\theta = \frac{v^2}{rg} = \frac{r\omega^2}{g}}$$

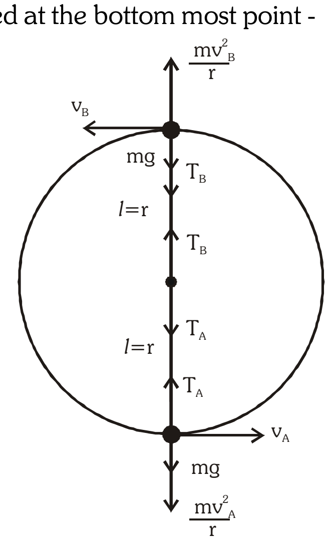
$T\cos\theta = mg$ $\boxed{T = \frac{mg}{\cos\theta}}$

$T\sin\theta = mr\omega^2$ $T\sin\theta = ml\sin\theta\ \omega^2$ $\left(\sin\theta = \frac{r}{l} \Rightarrow r = l\sin\theta\right)$ $\boxed{T = ml\omega^2}$

CIRCULAR MOTION IN VERTICAL PLANE

MOTION OF A BODY ATTACHED WITH STRING

- A body is attached to a massless string and certain velocity is imparted at the bottom most point -
- From point A to B, applying law of conservation of energy -
  loss in kinetic energy = gain in Potential Energy



$\Delta K = mg \times 2r$ $\Delta K = 2mgr$

$\frac{1}{2}mv_A^2 - \frac{1}{2}mv_B^2 = 2mgr$

$\boxed{v_A^2 - v_B^2 = 4gr}$

$v_A^2 = v_B^2 + 4gr$

If v_B = min, then v_A = min

$v_{A(min)}^2 = v_{B(min)}^2 + 4gr$

- At bottom most point : $F_C = T_A - mg$ or $T_A = \frac{mv_A^2}{r} + mg$

  $\frac{mv_A^2}{r} = T_A - mg$

- At top most point : $\frac{mv_B^2}{r} = T_B + mg$

- For v_B=min, the tension must be zero at topmost point $T_B = 0$

$$\frac{mv^2_{B(min)}}{r} = 0 + mg \qquad v^2_{B(min)} = rg \qquad v_{B(min)} = \sqrt{rg}$$

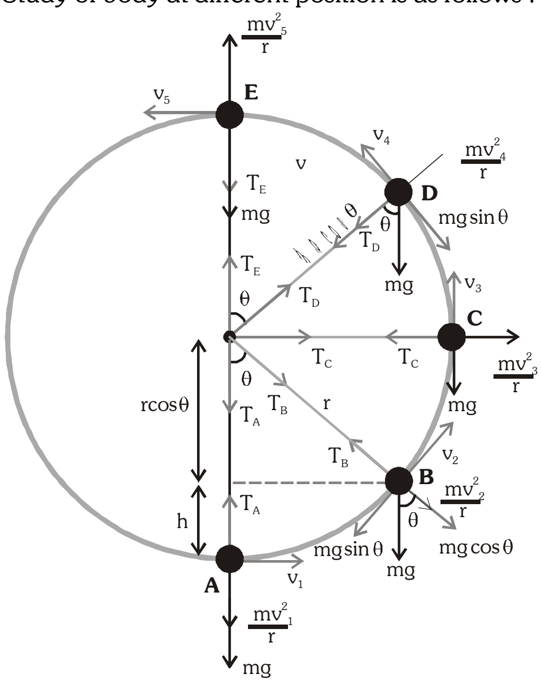
Therefore, $v^2_{A(min)} = v^2_{B(min)} + 4gr \qquad v^2_{A(min)} = gr + 4gr$

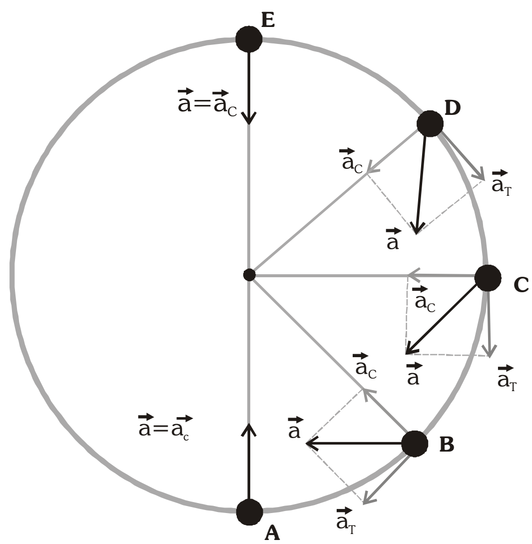
$$\boxed{v_{A(min)} = \sqrt{5gr}}$$

- This is the minimum velocity given at bottom so as to complete the circular path.

MASTER FIGURE FOR MOTION IN VERTICAL PLANE

- Study of body at different position is as follows :





- **At point A :** $\vec{F}_C = \frac{mv_1^2}{r} = T_A - mg$ $\qquad \boxed{T_A = \frac{mv_1^2}{r} + mg}$ maximum

$\vec{F}_T = 0 \qquad \vec{a}_C = \frac{v_1^2}{r}, a_T = 0 \qquad \vec{a} = \vec{a}_C$ (vertically upward)

- **At point B :** $\vec{F}_C = \frac{mv_2^2}{r} = T_B - mg\cos\theta \qquad T_B = \frac{mv_2^2}{r} + mg\cos\theta$

$\vec{F}_T = mg\sin\theta \qquad \vec{a}_C = \frac{v_2^2}{r}, a_T = g\sin\theta$

$\vec{a} = \vec{a}_C + \vec{a}_T$, $\vec{a}$ lie between $\vec{a}_C$ and $\vec{a}_T$

loss in kinetic energy =gain in P E =mgh $= mg(r - r\cos\theta) = mgr(1 - \cos\theta)$

- **At point C :** $F_C = \frac{mv_3^2}{r} = T_C \quad T_C = \frac{mv_3^2}{v}$

$F_T = mg \qquad a_C = \frac{v_3^2}{r}, a_T = g \qquad$ Loss in K. E. $=$ gain in P. E. $= mgr$

- **At point D :** $F_C = \frac{mv_4^2}{r} = T_D + mg\cos\theta$ $\quad T_D = \frac{mv_4^2}{r} - mg\cos\theta$

  $F_T = mg\sin\theta$ $\quad a_C = \frac{v_4^2}{r}, a_T = g\sin\theta$

- **At point E :** $F_C = \frac{mv_5^2}{r} = T_E + mg$ $\quad T_E + mg = \frac{mv_5^2}{r}$ $\quad \boxed{T_E = \frac{mv_5^2}{r} - mg}$ (minimum tension)

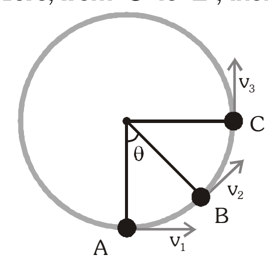
  $F_T = 0$ $\quad a_C = \frac{v_5^2}{r}, a_T = 0$

- At topmost point tension is minimum and at bottommost point tension is maximum, the maximum possibility of breaking of string is at point 'A'.
- The tension cannot be zero, from 'A' to just before 'C' , but the tension can be zero, from 'C' to 'E', therefore the string may slacken between 'C' and 'E'.

  It $v_1 = 0$, $\quad T_A = mg$

  It $v_2 = 0$, $\quad T_B = mg\cos\theta$

  It $v_3 = 0$, $\quad T_C = 0$



MOTION OF A BODY ATTACHED WITH RIGID ROD

- If the body is attached with a light rigid rod, The body will move as the rod rotates. According to COE

$$\frac{1}{2}mv_1^2 - \frac{1}{2}mv_2^2 = mg \times 2r$$

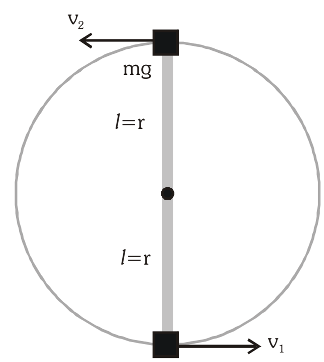
$$(v_1)^2 - (v_2)^2 = 4gr$$

$$(v_1)^2 = (v_2)^2 + 4gr$$

$$(v_1)^2_{min} = (v_2)^2_{min} + 4gr$$

$$(v_2)_{min} = 0$$

$$\boxed{(v_1)_{min} = \sqrt{4gr}}$$



- It is the minimum velocity given at bottom to complete the circle.

MOTION OF A BODY IN A VERTICAL CIRCULAR TRACK

- If a body is moving on vertical circular track then according to COE -

$$\frac{1}{2}mv_1^2 - \frac{1}{2}mv_2^2 = mg \times 2r$$

$$(v_1)^2 - (v_2)^2 = 4gr$$

$$(v_1)^2 = (v_2)^2 + 4gr$$

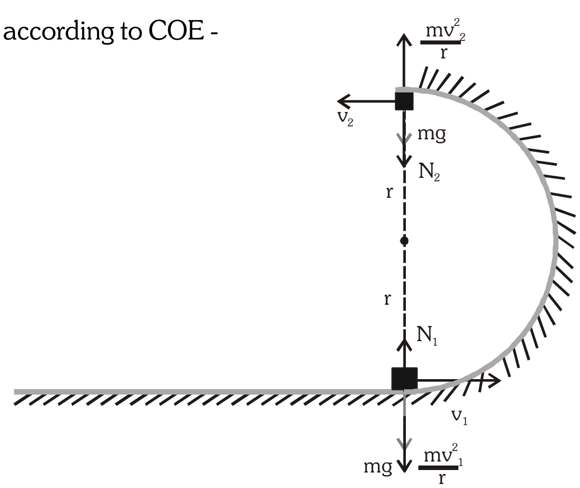
$$(v_1)^2_{min} = (v_2)^2_{min} + 4gr$$

- At top most point

$$\frac{mv_2^2}{r} = N_2 + mg$$

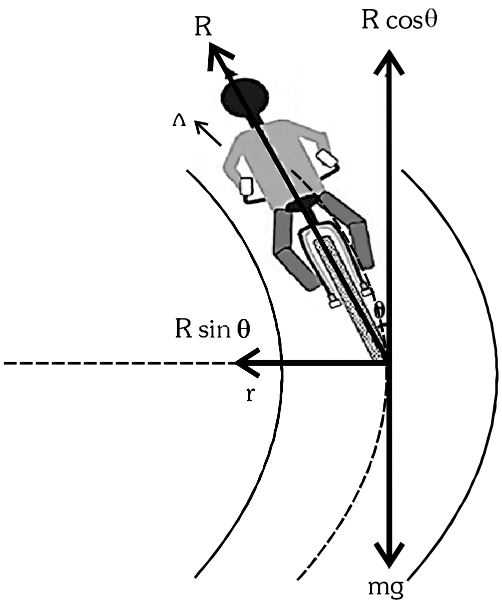
for v_2=min, $N_2 = 0$

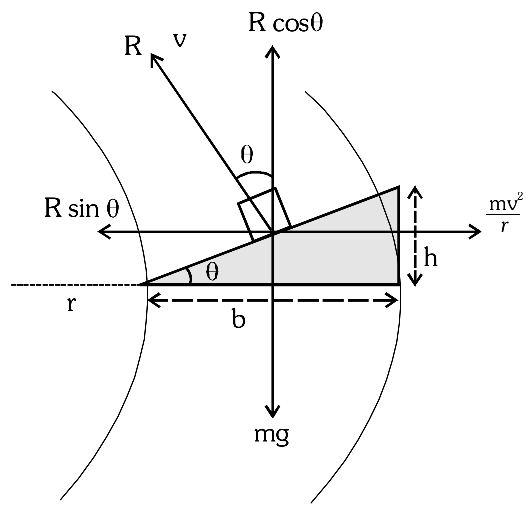
$$\frac{mv_2^2}{r} = 0 + mg \qquad v_2^2 = gr$$



$(v_2)_{min} = \sqrt{gr}$ Therefore, $(v_1)^2_{min} = gr + 4gr$

$\boxed{(v_1)_{min} = \sqrt{5gr}}$ This is the minimum velocity to complete the circle.





BENDING OF CYCLIST

- The cyclist bend at turn to overcome the arc by providing required centripetal force.

$$R\sin\theta = \frac{mv^2}{r} \quad \text{............ (i)}$$

$$R\cos\theta = mg \quad \text{............ (ii)}$$

$$\boxed{\tan\theta = \frac{v^2}{rg} = \frac{r\omega^2}{g}}$$

BANKING OF ROAD

- At turns roads are elevated from outer periphery so as to provide the required centripetal force.

$$R\sin\theta = \frac{mv^2}{r} \quad \text{............ (i)}$$

$$R\cos\theta = mg \quad \text{............ (ii)}$$

$$\boxed{\tan\theta = \frac{v^2}{rg} = \frac{r\omega^2}{g}} \qquad \boxed{\tan\theta = \frac{h}{b}}$$

θ = angle of banking

SKIDDING OF VEHICLE ON BANKED ROAD

CASE I

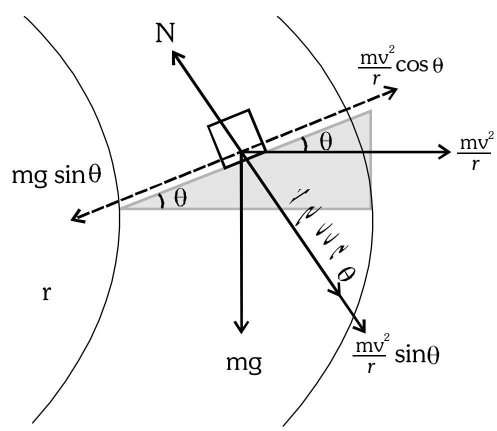
- If $\frac{mv^2}{r}\cos\theta = mg\sin\theta$,

there is no net force which makes vehicle to skid

$\boxed{f = 0}$

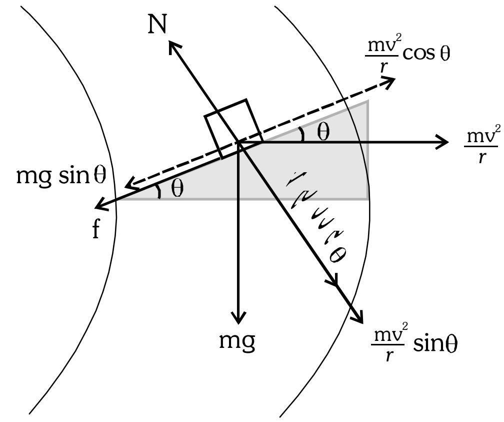
i.e. for no skidding $\frac{\cancel{m}v^2}{r}\cos\theta = \cancel{m}g\sin\theta$

$v^2 = gr\tan\theta$ $\boxed{v = \sqrt{gr\tan\theta}}$



CASE II

- If $\frac{mv^2}{r}\cos\theta > mg\sin\theta$, there is a net force directed radially outward and hence friction comes into existence inwardly i.e. $f \neq 0$[inwardly]



- For no skidding

$$\frac{mv^2}{r}\cos\theta = mg\sin\theta + f$$

- If v = max, f=f_L

$$\frac{mv^2_{max}}{r}\cos\theta = mg\sin\theta + f_L$$

$$\frac{mv^2_{max}}{r}\cos\theta = mg\sin\theta + \mu N$$

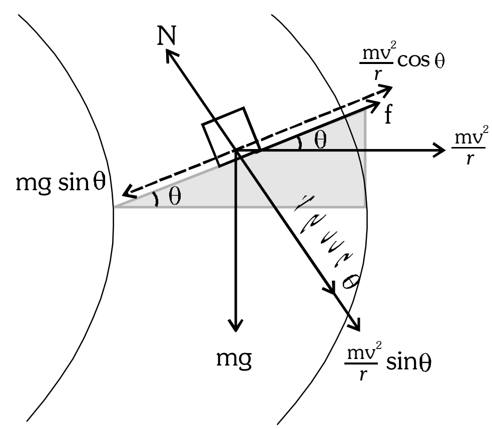
$$\frac{mv^2_{max}}{r}\cos\theta = mg\sin\theta + \mu\left[mg\cos\theta + \frac{mv^2_{max}}{r}\sin\theta\right]$$

$$\boxed{v_{max} = \sqrt{\frac{gr(\sin\theta + \mu\cos\theta)}{(\cos\theta - \mu\sin\theta)}}}$$

It is the maximum speed for no skidding

CASE III

- If $\frac{mv^2\cos\theta}{r} < mg\sin\theta$, there is a net force directed radially inward and hence friction comes into existence outwardly i.e. $f \neq 0$[outwardly]



- For no skidding -

$$\frac{mv^2\cos\theta}{r} + f = mg\sin\theta$$

$$\frac{mv^2\cos\theta}{r} = mg\sin\theta - f$$

- If, f=f_L, v = min

$$\frac{mv^2_{min}\cos\theta}{r} = mg\sin\theta - f_L$$

$$\frac{mv^2_{min}\cos\theta}{r} = mg\sin\theta - \mu N$$

$$\frac{mv^2_{min}\cos\theta}{r} = mg\sin\theta - \mu\left[mg\cos\theta + \frac{mv^2_{min}\sin\theta}{r}\right]$$

$$\boxed{v_{min} = \sqrt{\frac{gr(\sin\theta - \mu\cos\theta)}{(\cos\theta + \mu\sin\theta)}}}$$

It is the minimum speed for no skidding.

SKIDDING OF VEHICLE ON LEVEL (UNBANKED) ROAD

- For no sliding

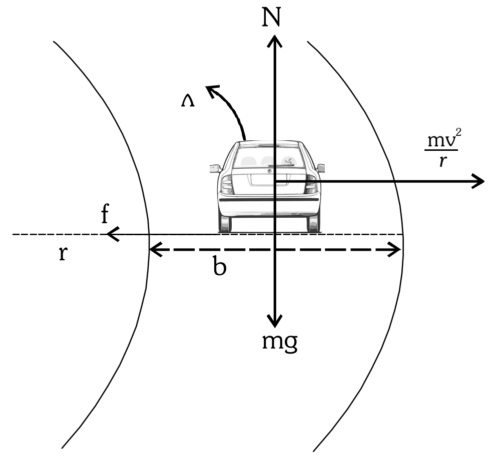
$$\frac{mv^2}{r} \le f_L$$

$$\frac{mv^2}{r} \le \mu N$$

$$\frac{mv^2}{r} \le \mu mg$$

$$v \le \sqrt{\mu rg}$$

$$\boxed{v_{max} = \sqrt{\mu rg}}$$



- It is the maximum speed for no skidding on road or minimum speed for skidding.

NOTES : -